8.4 Identifying Transformations

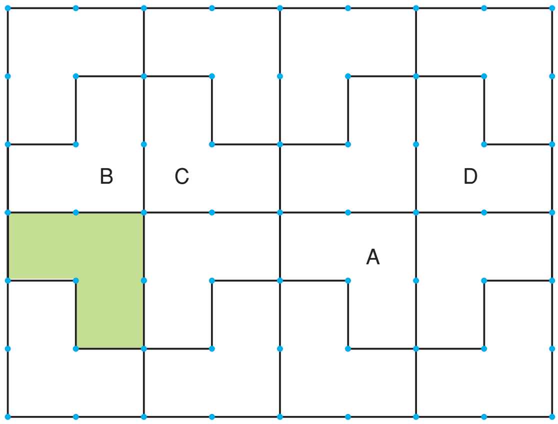
Focus Recognize transformation images.

Look around the classroom.
What transformations do you see?
How did you identify each transformation?

Investigate

Work with a partner.
Your teacher will give you a large copy of this design.

Tia used this design when she laid interlocking paving stones in her driveway.



To create the design, Tia translated, rotated, and reflected the shaded shape.
Each labelled shape is the image after a transformation.
Identify a transformation that produced each image.
Explain how you know.

Reflect & Share

Discuss your strategies for identifying each transformation with another pair of classmates.
How does the image relate to the original shape for each transformation?

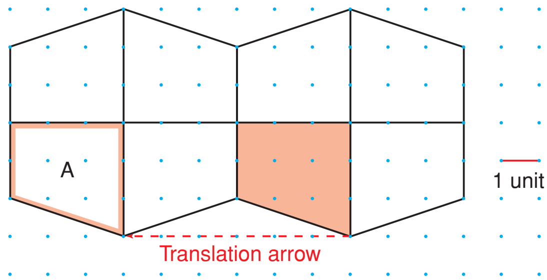
- a reflection
- a translation
- a rotation

Connect

Here is a design that shows 3 different transformations.

Translation

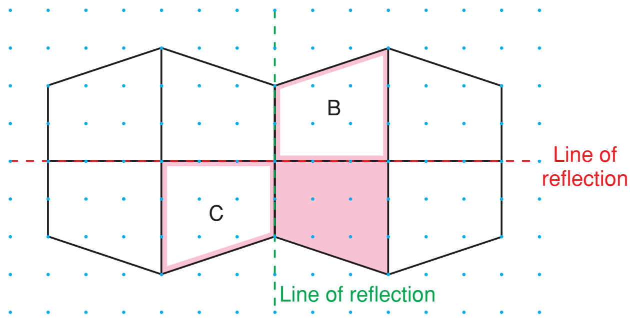
The shaded shape is translated 6 units left. Its translation image is Shape A. The translation arrow shows the movement in a straight line.
The translation image and the shaded shape are congruent and have the same orientation.



Reflection

The shaded shape is reflected in the red line of reflection.
Its reflection image is Shape B.

The shaded shape is reflected in the green line of reflection.
Its reflection image is Shape C.

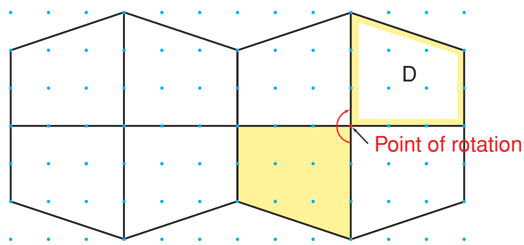


The shaded shape and each reflection image have opposite orientations.
Each reflection image and the shaded shape are congruent.

Rotation

The shaded shape is rotated 180° clockwise about the point of rotation.
The rotation image is Shape D.

We get the same image if the shaded shape is rotated 180° counterclockwise about the point of rotation.
The rotation image and the shaded shape are congruent and have the same orientation.



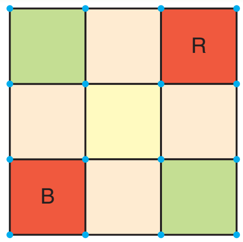
Under any transformation, the original shape and its image are always congruent.

Example 1

Look at this design of squares.
Describe each transformation.

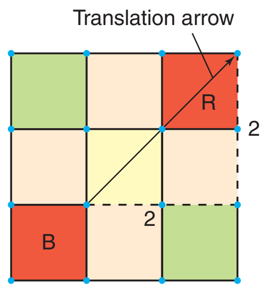
a) a translation for which Square R is an image of Square B

b) a reflection for which Square R is an image of Square B

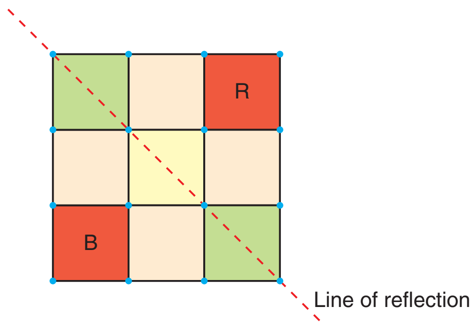


A Solution

a) Square R is the image of Square B after a translation 2 units right and 2 units up. The translation arrow shows the movement.



b) Square R is the image of Square B after a reflection in the slanted line. Use a Mira to verify the image.



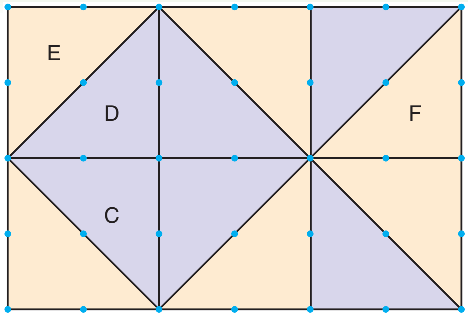
Example 1 shows that an image may be the result of more than one type of transformation.

Example 2

Look at this design of triangles.
Describe each rotation.

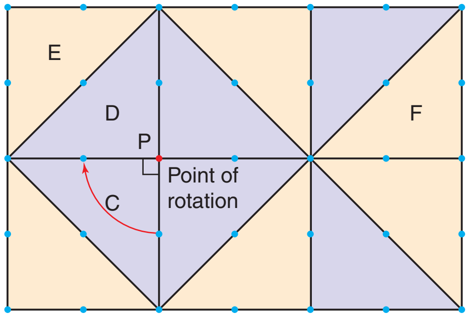
a) a rotation for which Triangle D is an image of Triangle C

b) a rotation for which Triangle F is an image of Triangle E

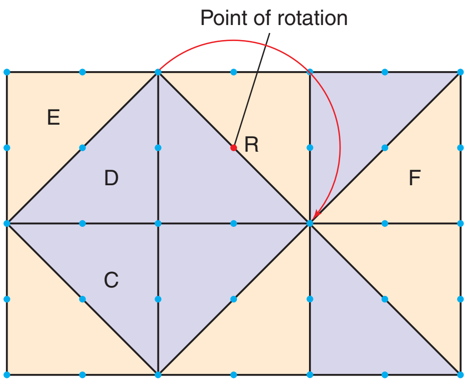


A Solution

a) Triangle D is the image of Triangle C after a rotation of 90° clockwise about P. P is a vertex the two triangles share. The same image is also the result of a rotation of 270° counterclockwise about P.



b) Triangle F is the image of Triangle E after a rotation of 180° about R. The point of rotation, R, is *not* on the shape being rotated.



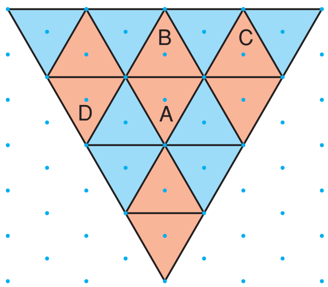
Discuss the ideas

1. In *Connect*, what does it mean when we say that a shape and its reflection image have opposite orientations?
2. In *Example 1b*, how is a point on the image related to a point on the original shape?
   How is the line segment that joins these points related to the line of reflection?
3. In *Example 1*, identify another transformation for which Square R is an image of Square B.
4. Can you change the size of a shape by translating, reflecting, or rotating it? Justify your answer.

Practice

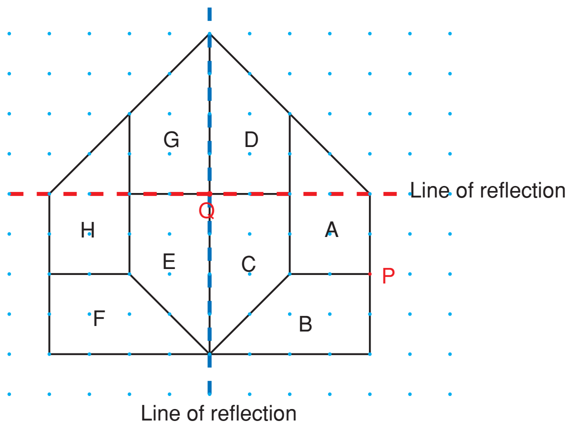
Check

5. In the design below, identify each transformation.



a) Shape B is the image of Shape A.
b) Shape C is the image of Shape A.
c) Shape D is the image of Shape A.
d) Shape C is the image of Shape B.

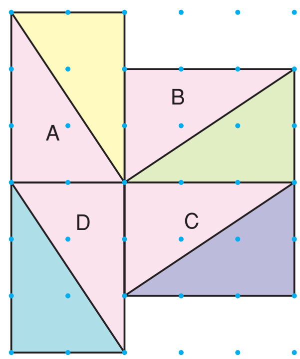
6. Use this design.



Match each transformation to a transformation image.
a) Rotate Shape A 90° counterclockwise about point P.
b) Reflect Shape C in the red line of reflection.
c) Translate Shape D 2 units right and 2 units down.
d) Rotate Shape G 180° about point Q.
e) Reflect Shape B in the blue line of reflection.

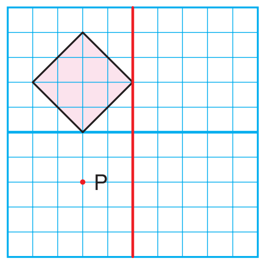
Apply

7. Identify each transformation.



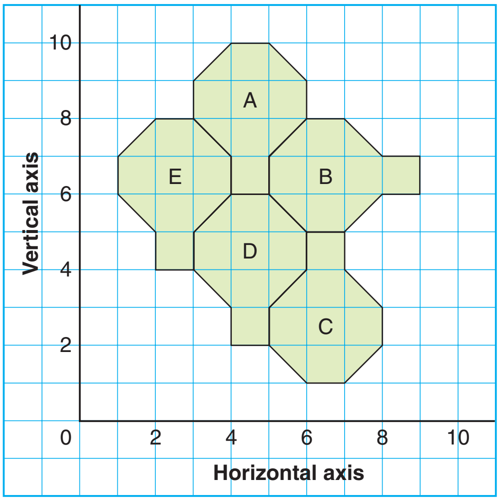
a) Shape A is the image of Shape B.
b) Shape B is the image of Shape C.
c) Shape C is the image of Shape D.
d) Shape D is the image of Shape A.

8. On grid paper, copy this square, the red and blue lines, and point P.



Draw the image of the original square after each transformation to create a design.
a) a translation 2 units right
b) a reflection in the red line
c) a rotation of 90° clockwise about P
d) a translation 2 units right and 4 units down
e) a reflection in the blue line

9. Assessment Focus How many different ways can each shape be described as a transformation of another shape? Explain.



10. Take It Further Use grid paper. In each case, describe the shape you drew.

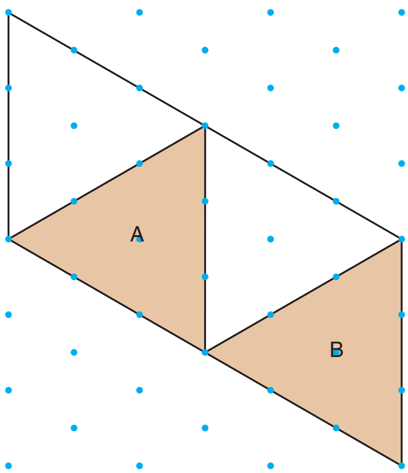
a) Draw a shape for which a translation image is also a reflection image and a rotation image. Draw the translation image.

b) Draw a shape for which a translation image is also a reflection image, but *not* a rotation image. Draw the translation image.

c) Draw a shape for which a translation image is *not* a reflection image *nor* a rotation image. Draw the translation image.

11. Take It Further This is a photo of a silk-screen print, *Haida Frog*, by Northwest Coast artist Bill Reid. Describe as many transformations in the print as you can. Include at least one translation, one reflection, and one rotation.

12. Take It Further Describe Shape A as a transformation image of Shape B in as many different ways as possible.



Reflect

When you see a shape and its transformation image in a design, how do you identify the transformation? Use diagrams in your explanation.

8.5 Constructing Tessellations

Focus Construct and analyse tessellations.

One of the basic ideas of Geometry is that of a *plane*. A plane is a flat surface. It has the property that a line joining any two points lies completely on its surface. What does this make you visualize?





Investigate

Work with a partner.
You will need tracing paper, plain paper, and a ruler.

One partner draws a triangle on his paper.
The other partner draws a quadrilateral on her paper.

Trace your shape.
Use the tracing to cover the paper with copies of your shape.
Try to do this with no overlaps or gaps.

You can rotate or flip the shape to try to make it fit.

Reflect & Share

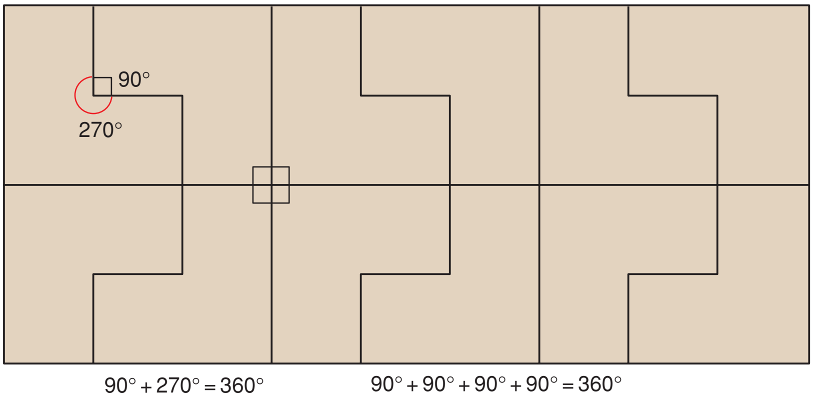
Compare your results with those of other classmates.
Can congruent copies of any triangle cover a plane with no overlaps or gaps? Justify your answer.
Can congruent copies of any quadrilateral cover a plane with no overlaps or gaps? Justify your answer.
What do you notice about the sum of the angles at a point where vertices meet?

Connect

When congruent copies of a shape cover a plane with no overlaps or gaps, we say the shape **tessellates**.
The design created is called a **tessellation**.

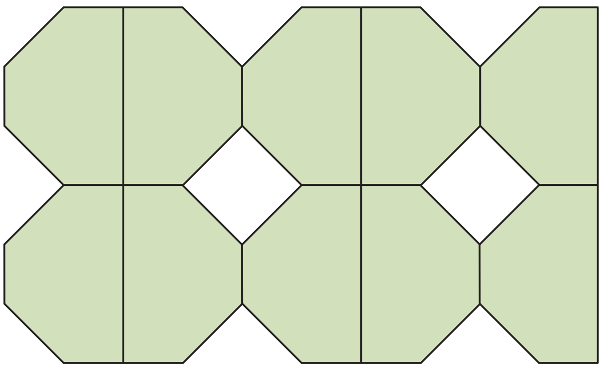
Not all polygons tessellate.

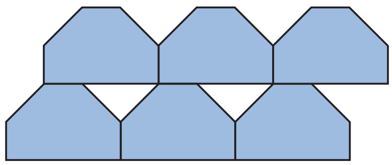
➤ This hexagon *does* tessellate.



➤ This hexagon *does not* tessellate.
Here are two different pictures to illustrate this.

There are gaps among the hexagons.



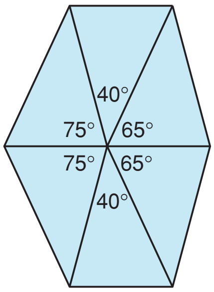


For copies of a polygon to tessellate, the sum of the angles at any point where vertices meet must be 360°. We say the *polygons surround a point*.

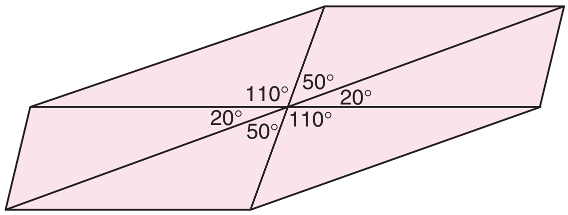
In *Investigate*, you found that triangles and quadrilaterals tessellate.

At any point where vertices meet, the sum of the angle measures is 360°.

Acute triangle



Obtuse triangle

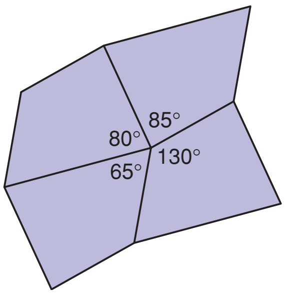


Six congruent triangles surround a point.

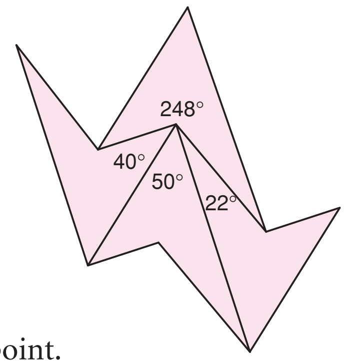
At each point:
75° + 40° + 65° + 65° + 40° + 75°
= 360°

At each point:
20° + 50° + 110° + 20° + 50° + 110°
= 360°

Convex quadrilateral



Concave quadrilateral



Four congruent quadrilaterals surround a point.

At each point:
80° + 85° + 130° + 65° = 360°

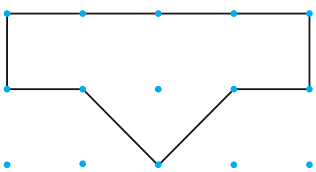
At each point:
50° + 40° + 22° + 248° = 360°

It is also possible for combinations of shapes to tessellate.

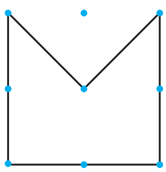
Example 1

Does each shape tessellate?
Justify your answer.

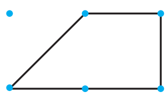
a)



b)



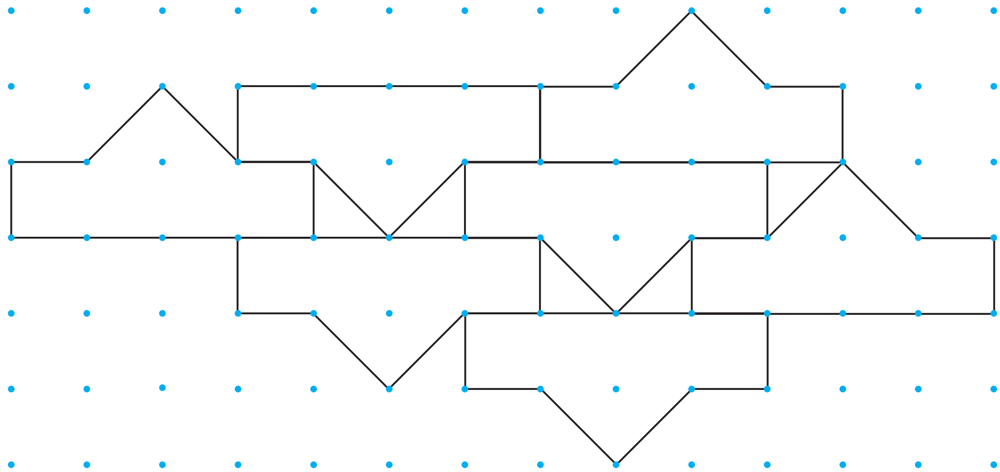
c)



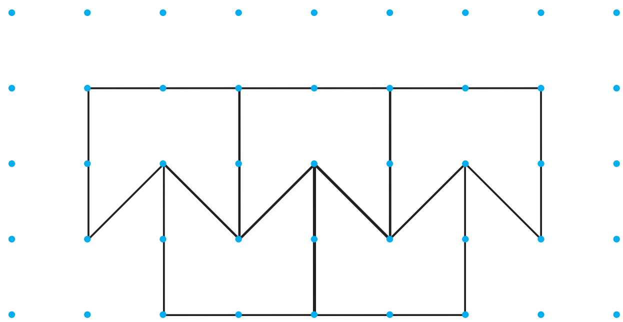
A Solution

Trace each shape.
Try to cover a page with the shape so there are no gaps or overlaps.

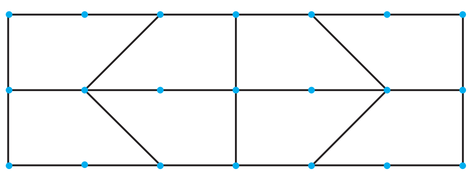
a) The shape does not tessellate.
There are gaps that are triangles.



b) The shape does tessellate.
There are no gaps or overlaps.



c) The shape does tessellate.
There are no gaps or overlaps.

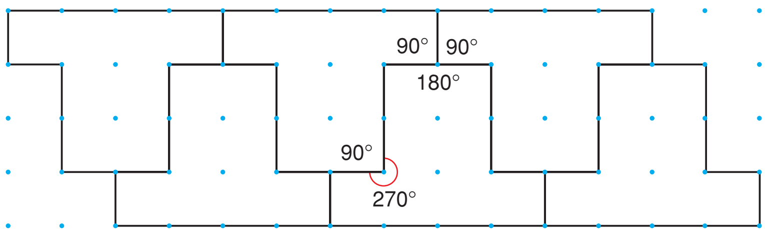


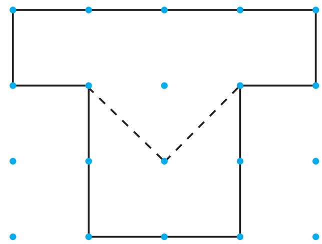
Example 2

Look at each shape in *Example 1* that does not tessellate.
Try to combine it with other shapes in *Example 1* so that
the combined shape tessellates.
How many ways can you do this?

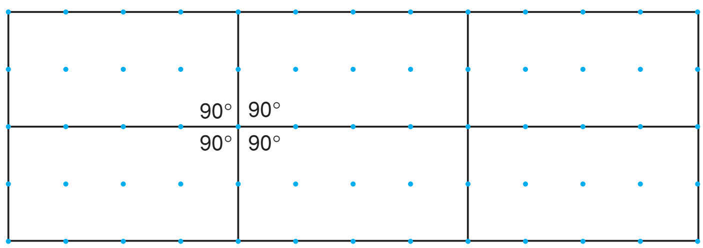
A Solution

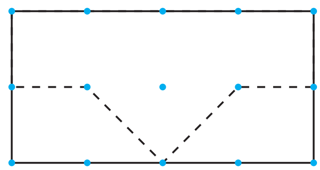
The shapes in parts a and b can be combined to make a T shape.
Tessellate with this new shape.





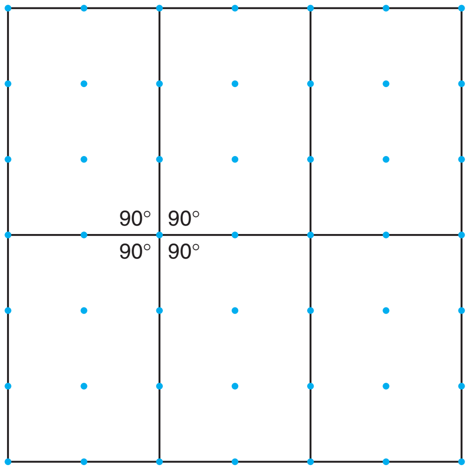
$90° + 90° + 180° = 360°$
$90° + 270° = 360°$
At each point where vertices meet, the sum of the angle measures is 360°.
So, the T shape tessellates.
The shape in part a and 2 of the shapes in part c can be combined
to make a rectangle.

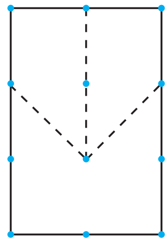




$90° + 90° + 90° + 90° = 360°$
At each point where vertices meet, the sum of the angle measures is 360°.
So, the rectangle tessellates.

The shape in part b and 2 of the shapes in part c can be combined
to make a rectangle.





$90° + 90° + 90° + 90° = 360°$
At each point where vertices meet, the sum of the angle measures is 360°.
So, the rectangle tessellates.

Examples 1 and *2* illustrate that a shape that does *not* tessellate may be combined with one or more shapes to make a new shape that tessellates. This new shape is called a **composite shape**.
In the *Practice* questions, you will investigate to find which other shapes and combinations of shapes tessellate.

Discuss the ideas

1. What is meant by “a shape tessellates”?
2. How can you tell that a shape does not tessellate?
3. Look around the classroom. Where do you see examples of tessellations? Describe each tessellation you see.
4. How do you know that all triangles tessellate?
5. How do you know that all quadrilaterals tessellate?

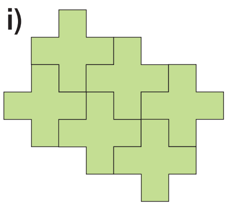
Practice

Check

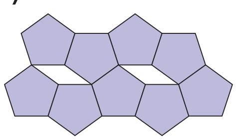
Keep all the tessellations you create for Lesson 8.6.

6. a) Which of these designs are tessellations? Justify your answer.

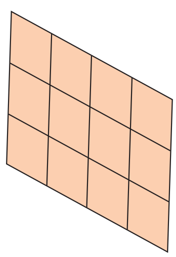
i)



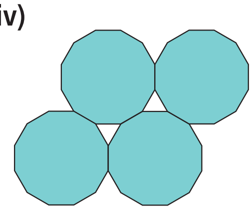
ii)



iii)

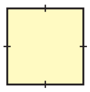


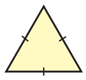
iv)

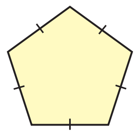


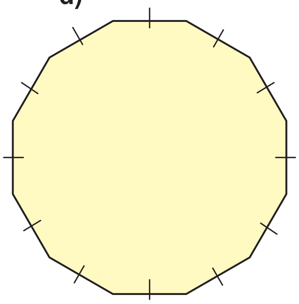
b) Which designs in part a are *not* tessellations? Justify your answer.

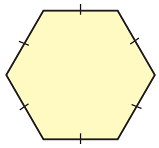
7. Use copies of each polygon. Does the polygon tessellate? If your answer is yes, create the tessellation. If your answer is no, explain how you know the shape does not tessellate.

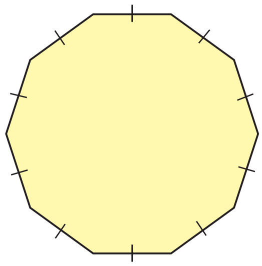
a) 

b) 

c) 

d) 

e) 

f) 

8. **a)** Use angle measures to justify your answers in question 7.

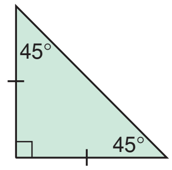
b) Which regular polygons tessellate? How do you know?

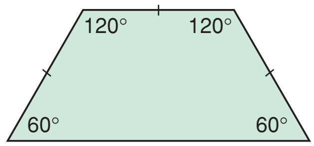
Apply

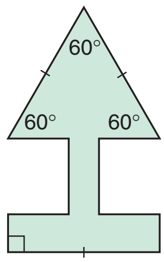
9. Use copies of the polygons in question 7. Find polygons that combine to make a composite shape that tessellates. How many different composite shapes can you find? Show your work.

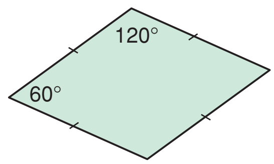
10. Use Pattern Blocks. Does the composite shape of a square and a regular hexagon tessellate? If your answer is yes, show the tessellation. If your answer is no, explain why not.

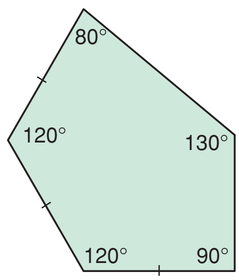
11. Use copies of each polygon. Which of these irregular polygons tessellate? Use angle measures to justify your answer.

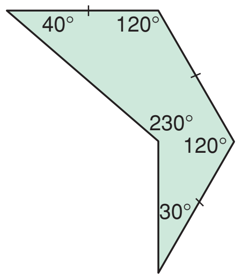
a) 

b) 

c) 

d) 

e) 

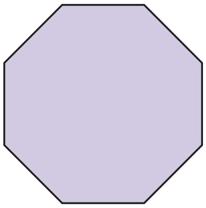
f) 

12. Assessment Focus

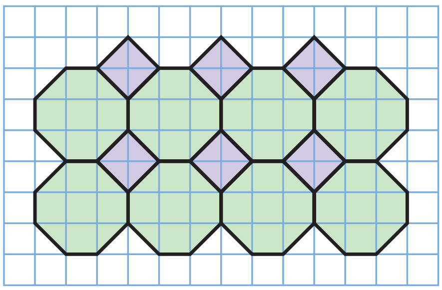
a) Which polygons in question 11 combine to make a composite shape that tessellates? Justify your answer in 2 ways.

b) How many different composite shapes can you find that tessellate? Show your work.

13. Here is a regular octagon.
Trace this octagon.
Does this octagon tessellate? Justify your answer.



14. Identify the two shapes that combine to make the composite shape in this tessellation. Explain how you know that this composite shape tessellates.



15. Where have you seen tessellations outside the classroom? What shapes were used in each tessellation?

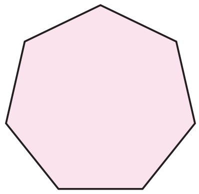
16. Take It Further In question 13, you discovered that a regular octagon does not tessellate. Use dot paper. Draw an octagon that tessellates. Explain why it tessellates.

17. Take It Further A 7-sided shape is called a heptagon.

a) Does a regular heptagon tessellate? Justify your answer.

b) If your answer to part a is yes, draw the tessellation.

c) If your answer to part a is no, draw a heptagon that tessellates.

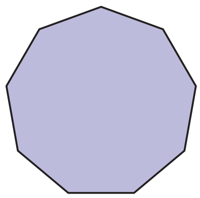


18. Take It Further A 9-sided shape is called a nonagon.

a) Does a regular nonagon tessellate? Justify your answer.

b) If your answer to part a is yes, draw the tessellation.

c) If your answer to part a is no, draw a nonagon that tessellates.



Reflect

How can you tell if a polygon or a composite shape tessellates?
Use examples in your explanation.

8.6 Identifying Transformations in Tessellations

Focus Create and analyse tessellations using transformations.

In Lesson 8.5, you drew tessellations.
In this lesson, we will use transformations to describe some of these tessellations.

Investigate

Work on your own.
You will need isometric dot paper and Pattern Blocks.
Choose two Pattern Blocks that form a composite shape that tessellates.

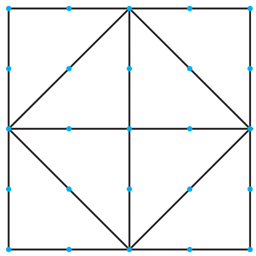
Make a tessellation to cover a plane.
Copy your tessellation onto dot paper.
Label each composite shape in your tessellation.
Explain the tessellation in terms of transformation images.
That is, how do you rotate, translate, or reflect each composite shape to create the tessellation? Write your instructions carefully.

Reflect & Share

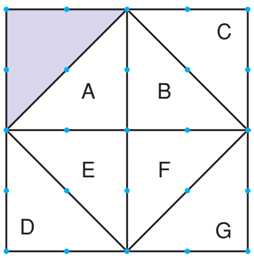
Trade instructions and the composite shape with a classmate.
Create your classmate's tessellation.
Check your tessellation with your classmate's tessellation.
How do they compare?

Connect

We can describe a tessellation in terms of transformations.

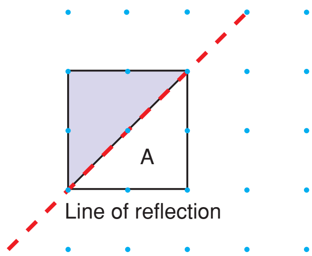


Label the shapes in the tessellation.

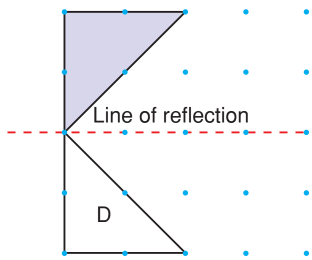


Start with the shaded shape.

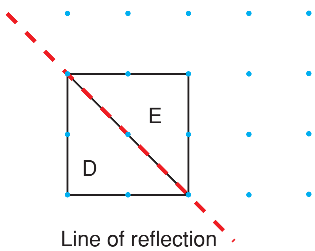
Step 1 To get Shape A, reflect the shaded shape in the line of reflection shown.



Step 2 To get Shape D, reflect the shaded shape in the line of reflection shown.



Step 3 To get Shape E, reflect Shape D in the line of reflection shown.



Repeat similar reflections to get shapes B, C, F, and G.

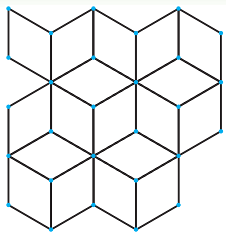
Under each transformation, the area of the shape does not change. This is known as **conservation of area**. This means that all the triangles in the tessellation have the same area.

Sometimes, a tessellation may be described by more than one type of transformation.

Example 1

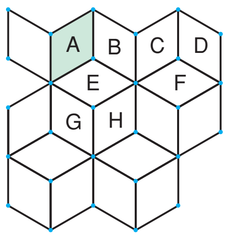
In this tessellation, identify:

a) a translation
b) a reflection
c) a rotation

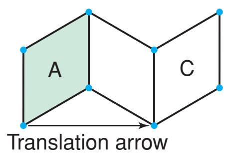


A Solution

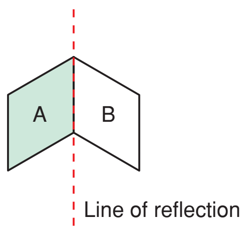
Label and shade Shape A as the original shape. Label some more shapes.



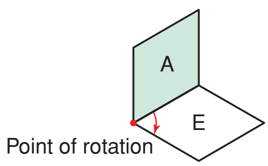
a) Shape C is a translation image of Shape A. Shape A has been translated right to get Shape C.



b) Shape B is a reflection image of Shape A. The line of reflection is the side the two shapes share.

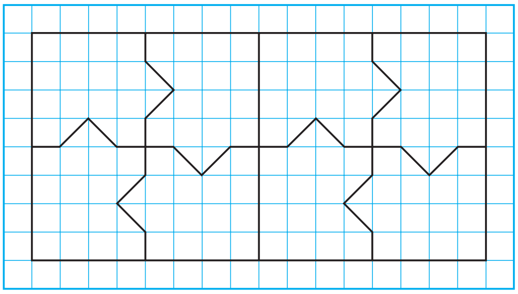


c) Shape E is a rotation image of Shape A. The point of rotation is a vertex they share. Shape A is rotated 60° clockwise or 300° counterclockwise to its image, Shape E.



Example 2

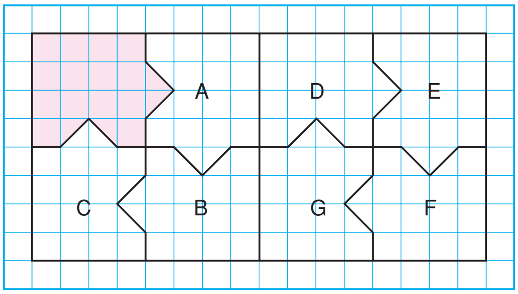
a) Identify a combination of transformations in this tessellation.



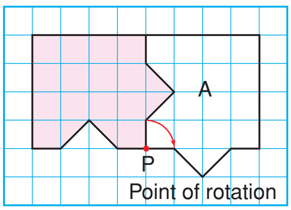
b) How do you know that the area of each shape is conserved?

A Solution

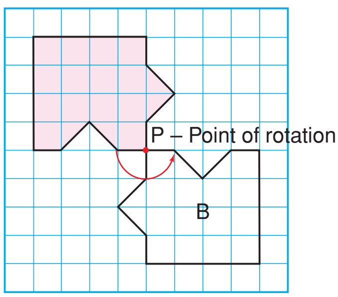
a) Label the shapes in the tessellation.



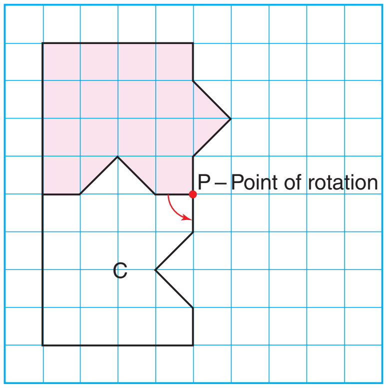
Start with the shaded shape.
To get Shape A, rotate the shaded shape 90° clockwise about P.



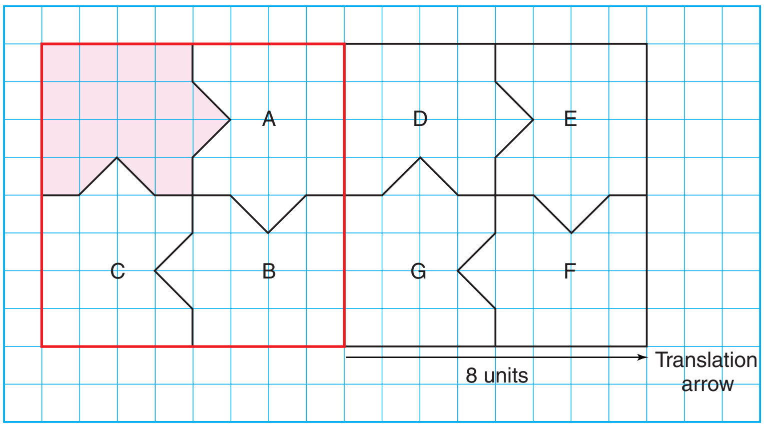
To get Shape B, rotate the shaded shape 180° about P.



To get Shape C, rotate the shaded shape 90° counterclockwise about P.



To get shapes D, E, F, and G, translate the square that contains the shaded shape, and shapes A, B, and C 8 units right.



b) Each image is congruent to the original shape.
Congruent shapes are identical.
So, the area of each image is equal to the area of the original shape.
This means that the area of each shape is conserved.

Discuss the ideas

1. Suppose the area of the shaded shape in *Example 1* is 8 cm^2. What is the area of Shape B? Shape E? How do you know?

2. In *Example 2*, identify a different combination of transformations in the tessellation.

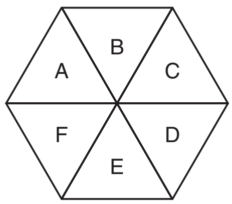
Practice

Check

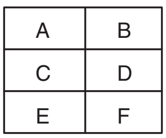
3. In each tessellation, Shape A is the original shape. In each tessellation, identify:

a) a translation
b) a reflection
c) a rotation

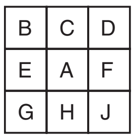
i)



ii)



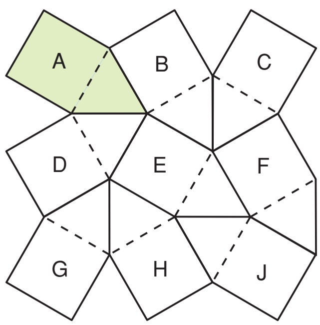
iii)



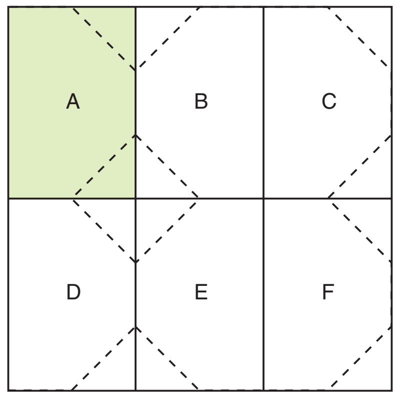
4. In each tessellation, Shape A is the original composite shape. In each tessellation, identify:

a) a translation
b) a reflection
c) a rotation

i)

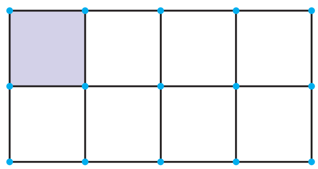


ii)

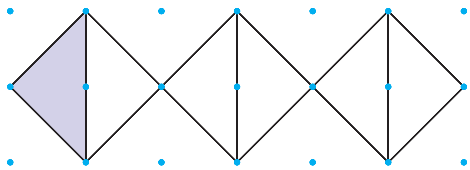


5. Here are three patterns. Describe the transformations that can be used to create each pattern. Start with the shaded shape.

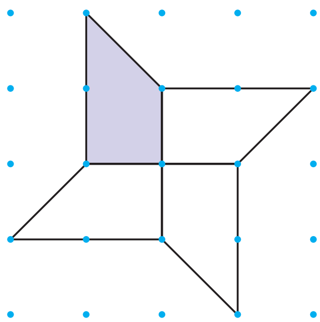
a)



b)

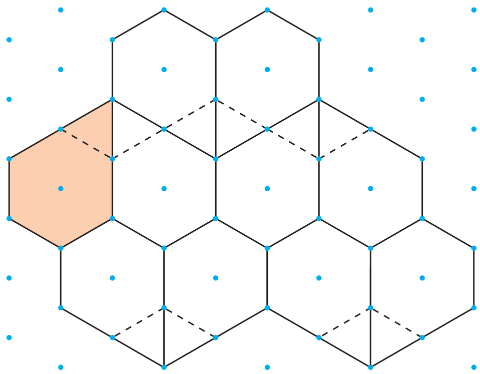


c)



Apply

6. Here is a tessellation one student drew for *Practice* question 9 in Lesson 8.5.



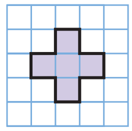
a) Describe the tessellation in terms of translations.

b) Describe the tessellation in terms of reflections.

c) Is area conserved when each shape is transformed? How do you know?

7. Look at the picture called *Knights on Horseback*, by M.C. Escher. Identify different transformations that may have been used to create the tessellation. Start with the red shape.

8. Use this shape and transformations to create a tessellation on grid paper.



Describe the tessellation in terms of transformations and conservation of area.

9. Here is a quilt design. Use a copy of the design.

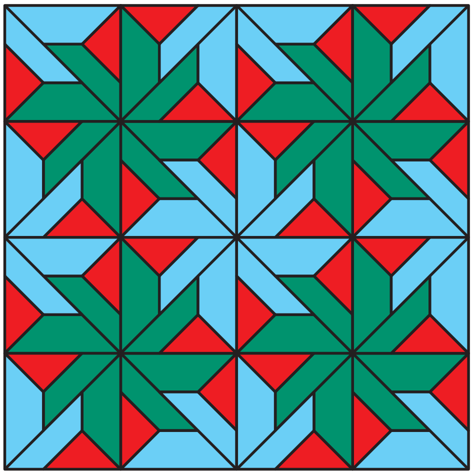
Find as many transformations in the design as you can. Ignore the different patterns on the material. Consider only the shapes.

10. Use transformations to create your own quilt design. Describe the transformations you used.

11. Choose 2 more tessellations you created in Lesson 8.5. Describe each tessellation in terms of transformations and conservation of area.

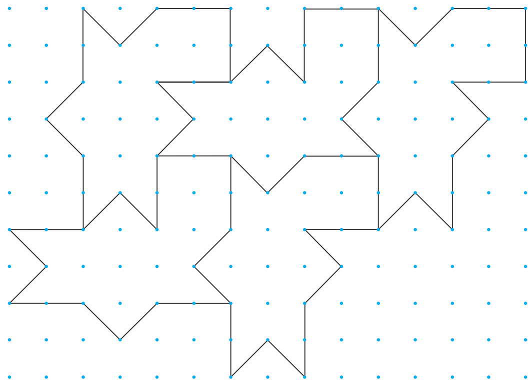
12. **Assessment Focus** Look at the tessellations you created in Lesson 8.5. Choose a tessellation that can be described in more than one way. Copy the tessellation onto dot paper. Label the shapes. Colour the tessellation. Describe the tessellation in terms of transformations and conservation of area. Describe the tessellation in as many ways as you can.

13. **Take It Further** Here is a flooring pattern. Use a copy of this pattern. Use transformations to describe the patterns in one square.



14. **Take It Further** The Alhambra is a walled city and fortress in Granada, Spain. It was built in the 14th century.

Here is part of one of its many tiling patterns. The pattern is a tessellation. Copy this tessellation onto dot paper.



a) Identify the composite shape that tessellates.

b) Continue the tessellation to cover the page.

c) Use transformations to describe the tessellation.

Reflect

When you use transformations to describe a tessellation, how do you decide which transformations to use? Include a tessellation in your explanation.

Target Tessellations

Players score points for each tessellation they create.
The first player to score 18 points wins.

HOW TO PLAY

1. Take turns to use the software to construct a shape that you think will tessellate.
   Use the software to check.

2. If the shape tessellates, the player scores 1 point for each side of the shape.
   For example, a 5-sided shape scores 5 points.
   If the shape does not tessellate, no points are scored.

3. Use the software to construct a different shape that you think will tessellate.
   The shape cannot have the same number of sides as the shape you used in *Step 1*.
   Use the software to check.

4. Play continues with a different shape each time.
   The first player to score 18 points wins.

TAKE IT FURTHER

Play the game again.
This time you choose the target number.
You cannot use shapes that have been used before.

YOU WILL NEED

Computer with geometry software, paper and pencil

NUMBER OF PLAYERS

2

GOAL OF THE GAME

To hit the target number exactly

What strategies did you use to hit the target number exactly?